Modave Lectures on Quantum Information
Problem Set

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The problems here will likely take much longer to solve than a single 50-minute lecture block. I recommend trying out the first two problems, which are relatively straightforward, and then at least reading through Problems 3 and 4, which are much more involved. Of course, I recommend working through these problems at your own pace afterwards; they are very illuminating!

Problem 1: Additivity of relative entropy

Show that $D(\rho_A \otimes \chi_B \,\|\, \sigma_A \otimes \tau_B) = D(\rho_A \,\|\, \sigma_A) + D(\chi_B \,\|\, \tau_B)$. You can assume that σ_A and τ_B are full-rank (no zero eigenvalues) to avoid divergences in relative entropy.

Note: If we think of D as a measure of distinguishability, then the result above is clear. The uncorrelated states in B cannot influence the distinguishability of the states of A and vice-versa. This could also be viewed as a special case of monotonicity of relative entropy, $D(\rho_{AB} \,\|\, \sigma_{AB}) \geq D(\rho_A \,\|\, \sigma_A)$.

Problem 2: Operator-sums are quantum channels

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces with $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$. Consider the linear map $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ whose action is

$$\mathcal{N}(X_A) = \sum_{j=1}^{d} N_j X_A N_j^\dagger, \tag{1}$$

where $d \leq d_A d_B$ and, for each $1 \leq j \leq d$, $N_j : \mathcal{H}_A \to \mathcal{H}_B$ and $\sum_{j=1}^{d} N_j^\dagger N_j = I_A$. Show that $\mathcal{N}$ is completely positive and trace-preserving.

Problem 3: Distance measures

Adapted from Exercise 2.7 of J. Preskill, Lecture Notes for Ph219/CS219: Quantum Information and Computation*, Chapter 2 (2013 edition).*

In many cases, we would like to be able to meaningfully quantify how "close" two quantum states are to each other. For example, if we are trying to correct errors made during a quantum computation, we would like to quantify how close the post-recovery state is to the original state. In this problem, we will see why the 1-norm is a good measure of closeness.

Consider two quantum states described by density operators ρ and $\tilde{\rho}$ in a N-dimensional Hilbert space, and consider the complete orthogonal measurement $\{E_a : a = 1, 2, \ldots, N\}$, where the E_a's are one-dimensional projectors satisfying

$$\sum_{a=1}^{N} E_a = I. \tag{2}$$

When the measurement is performed, outcome a occurs with probability $p_a = \mathrm{Tr}\, \rho E_a$ if the state is ρ and with probability $\tilde{p}_a = \mathrm{Tr}\, \tilde{\rho} E_a$ if the state is $\tilde{\rho}$.

The (normalized) L^1 distance between the two probability distributions is defined as

$$d(p, \tilde{p}) \equiv \|p - \tilde{p}\|_1 \equiv \frac{1}{2} \sum_{a=1}^{N} |p_a - \tilde{p}_a|. \tag{3}$$

This distance is zero if the two distributions are identical, and attains its maximum value of one if the two distributions have support on disjoint sets.

a) Show that

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i|, \tag{4}$$

where the λ_i's are the eigenvalues of the Hermitian operator $\rho - \tilde{\rho}$. *Hint:* Working in the basis in which $\rho - \tilde{\rho}$ is diagonal, find an expression for $|p_a - \tilde{p}_a|$, and then find an upper bound on $|p_a - \tilde{p}_a|$. Finally, use the completeness property Eq. (2) to bound $d(p, \tilde{p})$.

b) Find a choice for the orthogonal projectors $\{E_a\}$ that saturates the upper bound Eq. (4).

Define a distance $d(\rho, \tilde{\rho})$ between density operators as the maximal L^1 distance between the corresponding probability distributions that can be achieved by any orthogonal measurement. From the results of (a) and (b), we have found that

$$d(\rho, \tilde{\rho}) = \frac{1}{2} \sum_{i=1}^{N} |\lambda_i|. \tag{5}$$

c) The trace norm, or Schatten 1-norm $\|A\|_1$ of an operator A is defined as

$$\|A\|_1 \equiv \mathrm{Tr}\left[(A^\dagger A)^{1/2}\right]. \tag{6}$$

How can the distance $d(\rho, \tilde{\rho})$ be expressed as the 1-norm of an operator?

Now suppose that the states ρ and $\tilde{\rho}$ are pure states $\rho = |\psi\rangle\langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$. If we adopt a suitable basis in the space spanned by the two vectors, and appropriate phase conventions, then these vectors can be expressed as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \qquad |\tilde{\psi}\rangle = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix}. \tag{7}$$

d) Express the distance $d(\rho, \tilde{\rho})$ in terms of the angle θ.

e) Express $\||\psi\rangle - |\tilde{\psi}\rangle\|^2$ (where $\|\cdot\|$ denotes the Hilbert space norm, i.e., the 2-norm $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$) in terms of θ, and by comparing with the results of (d), derive the bound

$$d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) \leq \||\psi\rangle - |\tilde{\psi}\rangle\|. \tag{8}$$

f) Why is $\||\psi\rangle - |\tilde{\psi}\rangle\|$ *not* a good measure of the distinguishability of the pure quantum states ρ and $\tilde{\rho}$? *Hint:* Remember that quantum states are *rays*.

Problem 4: Positivity of relative entropy

Adapted from Exercise 10.1 of J. Preskill, Lecture Notes for Ph219/CS219: Quantum Information and Computation*, Chapter 10 (2018 edition).*

a) Show that $\log x \leq x - 1$ for all positive real numbers, with equality if and only if $x = 1$.

b) The classical relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$H(p \,\|\, q) = \sum_x p(x) \left(\log p(x) - \log q(x)\right), \tag{9}$$

for distributions such that $p(x) = 0$ if $q(x) = 0$, and where the sum is over x such that $q(x) \neq 0$. Show that

$$H(p \,\|\, q) \geq 0, \tag{10}$$

with equality if and only if the distributions are identical. (*Hint:* apply the inequality from (a) to $\log(q(x)/p(x))$.)

c) The quantum relative entropy of the density operator ρ with respect to σ is

$$D(\rho \,\|\, \sigma) = \mathrm{Tr}\left[\rho \log \rho - \rho \log \sigma\right], \tag{11}$$

and it is well-defined provided ker $\sigma \subseteq$ ker ρ. Let $\{p_i\}$ denote the eigenvalues of ρ and $\{q_a\}$ denote the eigenvalues of σ. Show that

$$D(\rho \,\|\sigma) = \sum_i p_i \left(\log p_i - \sum_a D_{ia} \log q_a\right), \tag{12}$$

where D_{ia} is a doubly stochastic matrix. Express D_{ia} in terms of the eigenstates of ρ and σ. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

d) Show that if D_{ia} is doubly stochastic, then (for each i)

$$\log\left(\sum_a D_{ia} q_a\right) \geq \sum_a D_{ia} \log q_a, \tag{13}$$

with equality only if $D_{ia} = 1$ for some a.

e) Show that

$$D(\rho \,\|\, \sigma) \geq H(p \,\|r), \tag{14}$$

where $r_i = \sum_a D_{ia} q_a$.

f) Show that $D(\rho \,\|\, \sigma) \geq 0$, with equality if and only if $\rho = \sigma$.