Lecture 3 - Quantum Error Correction

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- QI Processing: protect against
  - decoherence (interactions w/ env.)
  - gate errors (imperfect unitary ops.)

$\rightarrow$ Quantum Error Correction
(Distinct from fault tolerance — won't discuss here)

Idea: encode smaller $\mathcal{H}_{code} \hookrightarrow \mathcal{H}_{phys}$ nonlocally
$\rightarrow$ nonlocal logical info protected from local errors

Example: rudimentary 3-qubit code
- encode a single qubit in 3 qubits

"logical zero" $|\bar{0}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ "logical one" $|\bar{1}\rangle := \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$

- can correct a single bit flip: X_1, X_2, or X_3
- e.g. sps. X_1 gets applied erroneously

$$X_1|\bar{0}\rangle = |100\rangle + |011\rangle \qquad X_1|\bar{1}\rangle = |100\rangle - |011\rangle$$

- notice: $|\bar{0}\rangle, |\bar{1}\rangle$ are eigenstates of Z_1Z_2, Z_2Z_3, eigenval $+1$

but
$$Z_1Z_2(|100\rangle \pm |011\rangle) = -(|100\rangle \pm |011\rangle)$$
$$Z_2Z_3(|100\rangle \pm |011\rangle) = +(|100\rangle \pm |011\rangle)$$

- similarly,

syndrome \ error	X_1	X_2	X_3
Z_1Z_2	-1	-1	$+1$
Z_2Z_3	$+1$	-1	-1

$\rightarrow$ can diagnose where bit flip occurred & correct!

• a single phase flip (Z_1, Z_2, or Z_3) is bad news...

e.g. $Z_1|\bar{0}\rangle = |000\rangle - |111\rangle = |\bar{1}\rangle$ $\qquad$ $Z_1|\bar{1}\rangle = |000\rangle + |111\rangle = |\bar{0}\rangle$

i.e. Z_1, Z_2, or Z_3 are a logical $\bar{X}$ operator! ← still too local

• logical $\bar{Z} = X_1X_2X_3$
→ more nonlocality

Example: 9-qubit Shor code

$$|\bar{0}\rangle = (|000\rangle + |111\rangle)^{\otimes 3} \qquad |\bar{1}\rangle = (|000\rangle - |111\rangle)^{\otimes 3}$$

• Here, logical $\bar{Z} = X_1X_2X_3$ (among other possibilities)
logical $\bar{X} = Z_1Z_4Z_7$
• As before, Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9 can detect a bit flip in each block
• Now we can detect a single phase flip
→ measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$
• $|\bar{0}\rangle$, $|\bar{1}\rangle$ are both +1 eigenstates
• e.g. suppose Z_5 applied erroneously

$$X_1X_2X_3X_4X_5X_6\left(Z_5\left(a|\bar{0}\rangle + b|\bar{1}\rangle\right)\right) = -Z_5\left(a|\bar{0}\rangle + b|\bar{1}\rangle\right)$$
$$X_4X_5X_6X_7X_8X_9\left(Z_5\left(a|\bar{0}\rangle + b|\bar{1}\rangle\right)\right) = -Z_5\left(a|\bar{0}\rangle + b|\bar{1}\rangle\right)$$

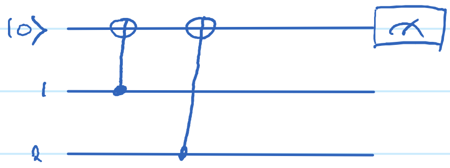
similarly

	Z_1 or Z_2 or Z_3	Z_4 or Z_5 or Z_6	Z_7 or Z_8 or Z_9
$X_1X_2X_3X_4X_5X_6$	−1	−1	+1
$X_4X_5X_6X_7X_8X_9$	+1	−1	−1

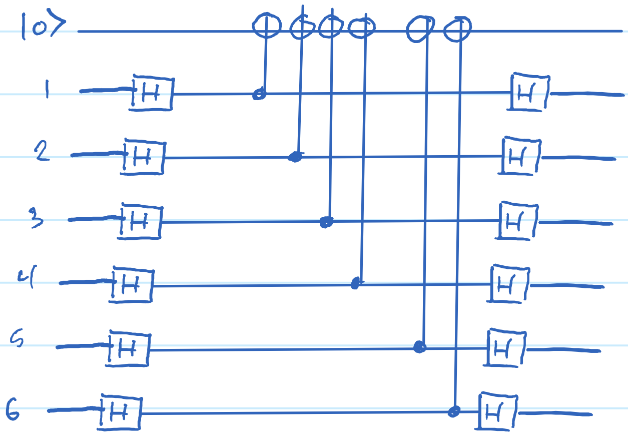
9-qubit Shor code encodes 1 logical qubit ($k=1$) in 9 physical qubits ($n=9$) & corrects (at least) 1 error
$\rightsquigarrow$ $[\![9,1,3]\!]$ code ($[\![n,k,d]\!]$)
↑ "distance", won't discuss here

Notes:
- can't correct arb. errors; most QECC's rely on noise being local, uncorrelated
- must avoid making local measurements!
- Q: how to measure collectively, nondestructively?
  ⇝ typically use ancillas

exercise:

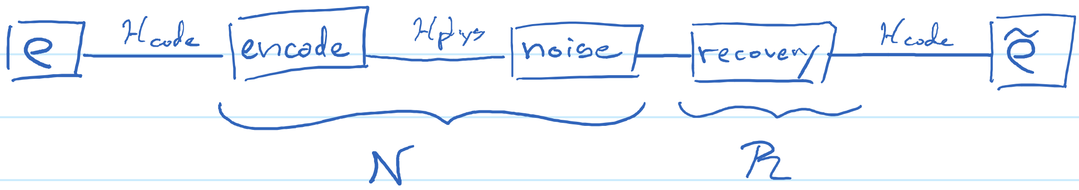


check that this measures $Z_1 Z_2$



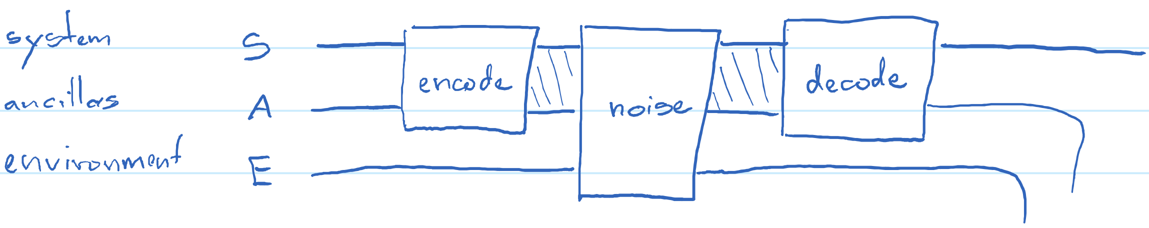
check that this measures $X_1 X_2 X_3 X_4 X_5 X_6$

QEC as a channel



$\tilde{\rho} = (\mathcal{R} \circ \mathcal{N})[\rho]$ ... hopefully $\tilde{\rho} \approx \rho$!

More detail:



Heuristic: for perfect recovery, final state of E can't depend on ρ_S

Q: When can you exactly reverse a channel?

Thm (Petz, Ohya) Let $N : S(\mathcal{H}_A) \to S(\mathcal{H}_B)$, $Q \subseteq S(\mathcal{H}_A)$. Then,

$D(\rho\|\sigma) = D(N[\rho]\|N[\sigma]) \quad \forall\ \rho, \sigma \in Q$ iff

$P_{\sigma,N}[\cdot] := \sigma^{1/2} N^\dagger\left[N[\sigma]^{-1/2}(\cdot)N[\sigma]^{-1/2}\right]\sigma^{1/2}$ exactly recovers ρ, σ.

$P_{\sigma,N}$: "Petz map"

- won't prove full version here (see Wilde Ch 12)
- trivial that $P_{\sigma,N}$ recovers σ
- check ρ for an easy version:

- Let $\dim \mathcal{H}_{code} = d < \infty$
- Isometrically embed $\mathcal{H}_{code} \overset{V}{\hookrightarrow} \mathcal{H}_{phys} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

  i.e. $V : \mathcal{H}_{code} \to \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

  $V^\dagger V = I_{code}$ $\quad VV^\dagger = \Pi^{(code)}$ ← projector onto $V\mathcal{H}_{code}$
- Let $N : S(\mathcal{H}_{code}) \to S(\mathcal{H}_A)$

  $\rho \mapsto \mathrm{Tr}_{\bar{A}}(V\rho V^\dagger)$ ("delete $\bar{A}$")
- Fix $\sigma \in S(\mathcal{H}_{code})$, let $\{|a\rangle_{code}\}_{a=1}^d$ be eigenbasis

  choose $\sigma = \sum_{a=1}^d \sigma_a |a\rangle\langle a|$ $\quad \sigma_a \neq 0$ (full-rank)

bookkeeping

- For exact recovery, will need $\mathcal{H}_A \simeq \mathcal{H}_1 \otimes \mathcal{H}_2 \oplus \mathcal{H}_3$

  where $\dim \mathcal{H}_1 = \dim \mathcal{H}_{code}$
- Then, choose basis of $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ s.t.

  $V|a\rangle_{code} = |a\rangle_1 \otimes |\chi\rangle_{2\bar{A}}$

  ↑ fixed $\forall\ a$

$$\begin{aligned}
N[\sigma] &= \mathrm{Tr}_{\bar{A}}\left[\sum_{a=1}^d \sigma_a |a\rangle\langle a|_1 \otimes |\chi\rangle\langle\chi|_{2\bar{A}}\right] \\
&= \left(\textstyle\sum_a \sigma_a |a\rangle\langle a|_1\right) \otimes \mathrm{Tr}_{\bar{A}} |\chi\rangle\langle\chi|_{2\bar{A}} \\
&\equiv \left(\textstyle\sum_a \sigma_a |a\rangle\langle a|_1\right) \otimes \chi_2 \\
&= \sigma_1 \otimes \chi_2
\end{aligned}$$

So $N[\sigma]^{-1/2} = \sigma_1^{-1/2} \otimes \chi_2^{-1/2}$

· write $\rho = \sum_{bc} \rho_{bc} |b\rangle\langle c|$

$$\Rightarrow \quad N[\rho] = \mathrm{Tr}_{\bar{A}}\left[\sum_{bc} \rho_{bc} |b\rangle\langle c|_1 \otimes |x\rangle\langle x|_{2A} \right]$$
$$= \rho_1 \otimes X_2$$

So $$N[\sigma]^{-1/2} N[\rho] N[\sigma]^{-1/2} = (\sigma^{-1/2} \rho \sigma^{-1/2})_1 \otimes (X^{-1/2} X X^{-1/2})_2$$
$$= (\sigma^{-1/2} \rho \sigma^{-1/2})_1 \otimes I_2$$

Q: action of N^+ ?

· let $\tau \in S(\mathcal{H}_{code})$

$$\langle N[\tau], \rho_1 \otimes I_2 \rangle_A = \mathrm{Tr}_A [(\tau_1 \otimes X_2)(\rho_1 \otimes I_2)]$$
$$= \mathrm{Tr}_1[\tau\rho]\, \mathrm{Tr}_2[X]$$
$$= \mathrm{Tr}_{code}[\tau\, (\mathrm{Tr} X\, \rho)]$$
$$= \langle \tau, (\mathrm{Tr} X)\rho \rangle_{code}$$

$\therefore \quad N^+[\rho_1 \otimes I_2] = \rho_{code} \cdot \mathrm{Tr} X$

$\therefore \quad P_{\sigma,N}[N[\rho]] = \sigma^{1/2} (\sigma^{-1/2} \rho \sigma^{-1/2}) \sigma^{1/2} \cdot \mathrm{Tr} X$
$$= \rho \cdot \mathrm{Tr} X$$

but notice, $\mathrm{Tr} X = \mathrm{Tr}_2 (\mathrm{Tr}_{\bar{A}} |x\rangle\langle x|_{2\bar{A}}) = \langle x | x \rangle_{2\bar{A}} = 1$

$\Rightarrow \quad P_{\sigma,N}[N[\rho]] = \rho$ success!

· check: $$D(N[\rho] \| N[\sigma]) = D(\rho_1 \otimes X_2 \| \sigma_1 \otimes X_2)$$
$$= D(\rho_1 \| \sigma_1)$$ ✓

Note: can do better - Universal Recovery Channel

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Thm $$R_{\sigma,N}[\cdot] = \int_{\mathbb{R}} dt\, \beta_0(t)\, \sigma^{-it/2} P_{\sigma,N}\left[N[\sigma]^{it/2} (\cdot) N[\sigma]^{-it/2} \right] \sigma^{it/2}$$

satisfies $D(\rho\|\sigma) - D(N[\rho] \| N[\sigma]) \geq -2 \log F(\rho, R \circ N[\rho])$

$F \equiv$ fidelity $\quad F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1 \quad \beta_0(t) = \frac{\pi}{2}(\cosh \pi t + 1)^{-1}$