Lecture 1 - Basics of Quantum Information

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What is Quantum Information Science?

Quantum mechanics + Information theory + computer Science

$\equiv$ info stored in and manipulation of quantum states

Classical info: bit strings $x_1 x_2 \cdots x_n$ $\quad x_i \in \{0,1\}$

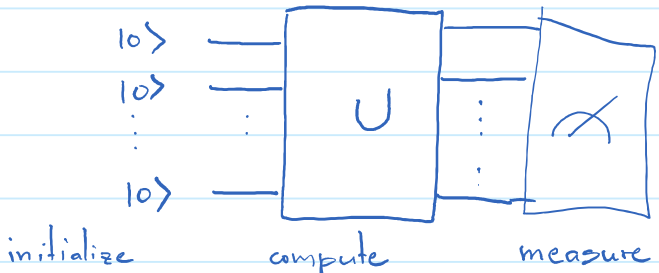
Quantum info: qubit strings $\displaystyle\sum_{x_1=0,1} \cdots \sum_{x_n=0,1} c_{x_1 \cdots x_n} |x_1\rangle \otimes \cdots \otimes |x_n\rangle$

Q: How is quantum info different?

- true randomness
- uncertainty
- no cloning
- entanglement
- superpositions ("quantum parallelism")

Q: What is a quantum computation?

State space
e.g. n qubits



output: sample prob. distro

$\Pr(x_1 x_2 \cdots x_n)$

$= |\langle x_1 x_2 \cdots x_n | U | 00 \cdots 0\rangle|^2$

$\rightsquigarrow$ hopefully, answer x to an interesting problem occurs w/ high prob.!

- won't discuss computation & algorithms in these lectures
- instead, mathematical properties of QI & how it evolves, w/ some applications to HEP
- central idea: Quantum Channel

  $\rightarrow$ most general map b/w quantum states

Outline: ① Basics of QI (states, entropy, black holes)

② Exercises: relative entropy & distinguishability

③ Channels

④ Quantum Error Correction

⑤ Holography as QEC

States

Let $\mathcal{H}$: Hilbert space, $\dim \mathcal{H} = d$, ON basis $\{|i\rangle\}_{i=1}^{d}$

countable ∞ ok

- generic pure state $|\psi\rangle \in \mathcal{H}$

$$|\psi\rangle = \sum_{i=1}^{d} c_i |i\rangle \qquad \langle\psi|\psi\rangle = \sum |c_i|^2 = 1$$

- generic mixed state $\rho \in \mathcal{L}(\mathcal{H})$ density matrix

$$\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$$

Properties of density matrices:

1. $\rho = \rho^\dagger$ (Hermitian)
2. $\langle\psi|\rho|\psi\rangle \geq 0$ (pos. semidef)
3. $\mathrm{Tr}\,\rho = 1$ (normalization)
4. $\exists$ basis $\{|a\rangle\}$: $\rho = \sum_{a=1}^{d} p_a |a\rangle\langle a|$, $p_a \geq 0$, $\sum p_a = 1$
5. If only one $p_a \neq 0$ $\rightarrow$ $\rho = |a\rangle\langle a|$ pure
6. $\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho\mathcal{O})$

Notation: $S(\mathcal{H}) \equiv$ set of density matrices on $\mathcal{H}$

Composition of Hilbert spaces

- $\mathcal{H}_A = \mathrm{span}\{|i\rangle_A\}_{i=1}^{d_A}$ $\qquad \mathcal{H}_B = \mathrm{span}\{|\mu\rangle_B\}_{\mu=1}^{d_B}$

$$\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathrm{span}\{|i\rangle_A \otimes |\mu\rangle_B\}_{i=1,\,\mu=1}^{d_A\ \ d_B}$$

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = d_A d_B$$

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \Rightarrow |\psi\rangle = \sum_{i=1}^{d_A}\sum_{\mu=1}^{d_B} c_{i\mu} |i\rangle_A \otimes |\mu\rangle_B$$

Def^n $\langle\mu|_B : \mathcal{H}_{AB} \to \mathcal{H}_A$ $\quad \langle\mu|_B (|i\rangle_A \otimes |\nu\rangle_B) = \delta_{\mu\nu} |i\rangle_A$

Def^n Partial trace $\mathrm{Tr}_B : \mathcal{L}(\mathcal{H}_{AB}) \to \mathcal{L}(\mathcal{H}_A)$

$$\mathrm{Tr}_B\, O_{AB} = \sum_{\mu=1}^{d_B} \langle\mu|_B\, O_{AB}\, |\mu\rangle_B$$

ex $\rho_{AB} = \sum_{i\mu} \sum_{j\nu} \rho_{i\mu j\nu} |i\rangle_A |\mu\rangle_B \langle j|_A \langle \nu|_B$

$$\mathrm{Tr}_B\, \rho_{AB} = \sum_\lambda \langle\lambda|_B \left(\sum_{i\mu}\sum_{j\nu} \rho_{i\mu j\nu} |i\rangle_A|\mu\rangle_B \langle j|_A \langle\nu|_B \right) |\lambda\rangle_B$$

$$= \sum_{i\mu}\sum_{j\nu} \rho_{i\mu j\nu} |i\rangle\langle j|_A \left(\sum_\lambda \underbrace{\langle\lambda|\mu\rangle}_{\delta_{\lambda\mu}} \underbrace{\langle\nu|\lambda\rangle_B}_{\delta_{\lambda\nu}} \right)$$

$$= \sum_{ij} \left(\sum_\mu \rho_{i\mu j\mu} \right) |i\rangle\langle j|_A$$

$$\equiv \rho_A \qquad \text{reduced state on } A$$

Entropy

Def^n Let $\rho \in S(\mathcal{H})$. The Von Neumann entropy of ρ is

$$S(\rho) = -\mathrm{Tr}\, \rho \log \rho$$

ex $\rho = \sum_i p_i |i\rangle\langle i|$ then $S(\rho) = -\sum_i p_i \log p_i$

- a measure of purity
- a measure of bipartite entanglement

// in fact, the unique measure of pure-state bipartite entanglement

Properties: (assume $\dim \mathcal{H} = d < \infty$ for now)

1. $\max_\rho S(\rho) = \log d$

Proof: work in eigenbasis of ρ : $S(\rho) = -\sum_{i=1}^{d} p_i \log p_i$

- Maximize $S(\rho) = S(p_1, \dots, p_d)$ subject to
  $0 \le p_i \le 1$, $\sum p_i = 1$

- let $p_d = 1 - \sum_{i=1}^{d-1} p_i$ (take care of last constraint)
- $\frac{\partial S}{\partial p_a} = -\log p_a \cancel{-1} - \left(-\log\left(1 - \sum_{i=1}^{d-1} p_i\right) \cancel{-1}\right)$

  $= \log p_d - \log p_a \overset{!}{=} 0$

$\Rightarrow$ $p_d = p_a \quad \forall\, a = 1, \dots, d-1$

$\Leftrightarrow$ $p_i = \frac{1}{d} \quad \forall\, i$

$\therefore$ $\max S(\rho) = \log d$, achieved on $\rho = \frac{I}{d}$ maximally mixed

2. $S(\rho) = 0 \Leftrightarrow \rho = |\psi\rangle\langle\psi|$ pure

   Proof: $\Rightarrow$ trivial

   $\Leftarrow$ above, we found a single critical point of $S(\rho)$, and it was a maximum

   $\therefore$ minimum must be achieved at an edge pt. $p_a = 1$, $p_{i \neq a} = 0$

   $\Rightarrow$ this is a pure state, for which $S(\rho) = 0$

$\rightsquigarrow$ $S(\rho)$ is a measure of purity

3. Let $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$ be pure, $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ $\rho_B = \mathrm{Tr}_A |\psi\rangle\langle\psi|$.

   Then $S(\rho_A) = S(\rho_B)$

   Follows from Schmidt decomposition / SVD

4. For $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$, $S(\rho_A) = S(\rho_B) = 0$ iff $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$,

   i.e. $|\psi\rangle_{AB}$ is unentangled across A, B

   $\rightarrow$ for bipartite ρ_{AB}, $S(\rho_A)$, $S(\rho_B)$ is sometimes called entanglement entropy

5. $S(U \rho U^\dagger) = S(\rho)$ for any unitary U

   $\rightarrow$ S is invariant under local operations

Entropy also obeys a lot of inequalities, e.g.

- Subadditivity: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$
  equality iff $\rho_A = \rho_A \otimes \rho_B$

- Araki - Lieb: $|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB})$

- Strong Subadditivity: $S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_C)$

... and more!

- there are many more entropic quantities that one can define/interpret
- an important one for us is relative entropy

Def^n Let $\rho, \sigma \in S(\mathcal{H})$. The relative entropy of ρ and σ is

$$D(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$$

☆ Key property ①: $D(\rho\|\sigma) \geq 0$
$D(\rho\|\sigma) = 0$ iff $\rho = \sigma$
(will prove in exercise session)

☆ Key property ②: $D(\rho\|\sigma) \geq \frac{1}{2\ln 2} \|\rho - \sigma\|_1^2$

(Pinsker's ineq. - won't prove; see Wilde)

Why is this important?
→ Data-processing inequality (next lecture)
→ $\|\rho - \sigma\|_1$ is a measure of the distinguishability of ρ, σ
(exercise session)

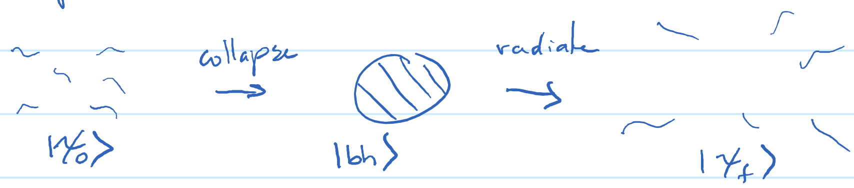
$\therefore$ $D(\rho\|\sigma)$ small $\Rightarrow$ ρ, σ are "close"

Def^n $\|O\|_1 = \mathrm{Tr}(\sqrt{O^\dagger O})$

The Black Hole Information Problem

Modern version, due to Almheiri, Marolf, Polchinsky, Stanford, Sully:
The following 4 postulates are inconsistent:

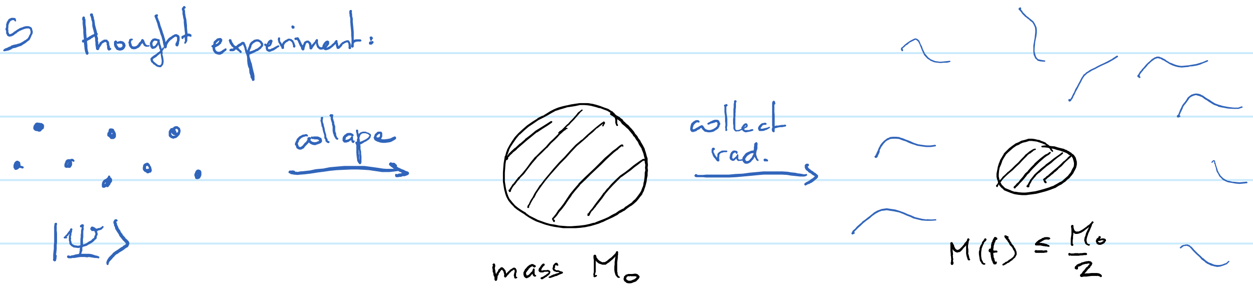
1. Unitarity

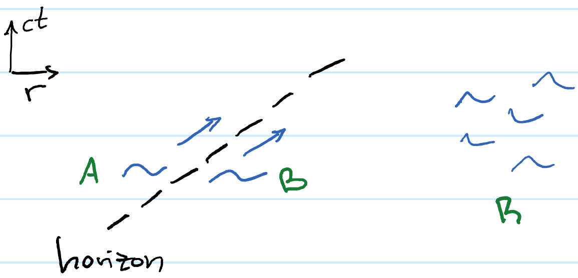


2. L.E.F.T. : outside bh stretched horizon, physics described by local effective field theory
3. Black holes are QM systems: discrete spectrum of states, density given by $S_{BH} = \frac{A}{4G_N}$

4. No Drama: nothing special (beyond semi-class GR + EFT) happens at stretched horizon

AMPS thought experiment:





A: outgoing mode inside hor.
B: " " outside "
R: "early" radiation

(i) SSA: $S_{AB} + S_{BR} \geq S_{ABR} + S_B$
(ii) unitarity $\Rightarrow$ $S_{BR} < S_R$ (B purifies R)
(iii) no drama $\Rightarrow$ $S_{AB} = 0$ (AB in pure near-vac. state)
$\Rightarrow$ $S_{ABR} = S_R$ (subadditivity)

so $S_R + S_B \overset{(iii)}{=} S_{ABR} + S_B \overset{(i)}{\leq} \cancel{S_{AB}}^{\,0} + S_{BR} \overset{(ii)}{<} S_R$ $\#$

∴ need to give something up

(i) Unitarity → bh destroy info
(ii) L.E.F.T → nonlocality
holography
(iii) BH are QM → remnants
→ beyond QM?
(iv) No Drama → firewalls ($S_{AB} \neq 0$)
→ fuzzballs